

# Programmable quantum state transfer

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A programmable quantum networks model is used in this paper for development of methods of control of a quantum state transport. These methods may be applied for a wide variety of patterns of controlled state transmission and spreading in quantum systems. The programmable perfect state transfer and quantum walk, mobile quantum (ro)bots and lattice gas automata may be described by unified way with such approach.

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## I. INTRODUCTION

Different kinds of the non-optical quantum state transport using specific phenomena in “quantum wires” are investigated very actively during recent few years. Some references may be found in Sec. IV and Sec. V. In the present work is discussed a compact theoretical approach to the programmable quantum state transfer. These methods have applications for the coherent control of quantum systems, the theory of quantum communications and computations. They also uncover promising relations between different models of the quantum information science.

The programmable state transfer is introduced as a particular kind of conditional quantum dynamics in Sec. II. Application of these methods to simple motion along lattices controlled by state of a qubit is discussed in Sec. III. A relation with programmable extension of so-called perfect quantum state transfer with lattices and spin chains is considered in the Sec. IV and some possibilities of the control of the coined quantum walk are described in the Sec. V.

## II. CONDITIONAL QUANTUM DYNAMICS

An essential part of quantum information processing, is the “*conditional quantum dynamics*, in which one subsystem undergoes coherent evolution that depends on the quantum state of another subsystem” [1].

A simple example of the conditional quantum dynamics may be written as [1]

$$C = \sum_k |k\rangle\langle k| \otimes U_k. \quad (1)$$

We can denote Hilbert spaces of these two subsystems as  $\mathcal{H}_P$  (“program” or a control system),  $\mathcal{H}_d$  (“data” or a target system) and to consider a gate  $G$  on  $\mathcal{H}_P \otimes \mathcal{H}_d$ . If we do not accept entanglement between two subsystems, most

general form of conditional or programmable evolution may be expressed as [2]

$$G : [|P_U\rangle \otimes |d\rangle] \rightarrow |P'_U\rangle \otimes (U|d\rangle), \quad (2)$$

where  $|d\rangle \in \mathcal{H}_d$  is an arbitrary state of the target system and  $|P_U\rangle \in \mathcal{H}_P$  is a state of the control system (“a program register”) implementing operator  $U$ .

In [2] was shown that if two states  $|P_\alpha\rangle$  and  $|P_\beta\rangle$  of the program register implement two different operators  $U_\alpha$  and  $U_\beta$ , then Eq. (2) implies

$$\langle P_\alpha | P_\beta \rangle = 0. \quad (3)$$

Due to Eq. (3) all states of the program register are orthogonal and the dimension of  $\mathcal{H}_P$  is equal to the number of different operators we need to implement. It was used in [2] as an inspiration to the theory of stochastic programmable quantum devices, but there are also implications to usual unitary evolution, discussed in the present paper.

Due to Eq. (3) we may without lost of generality to use states of the control register implementing different programs as a new computational basis  $|k\rangle$  [3]. In such a case the operator  $C$  Eq. (1) satisfies Eq. (2) for the basis states, i.e.,  $C[|k\rangle \otimes |d\rangle] = |k\rangle \otimes (U_k|d\rangle)$ . A possible change of the state of the control system in Eq. (2) may be described using the composition of  $C$  with  $A \otimes \mathbb{1}$ , i.e., an arbitrary unitary operator on the first subsystem.

For an arbitrary state  $|\psi\rangle = \sum_k \psi_k |k\rangle$  of the control register the operator Eq. (1) does not satisfy Eq. (2), because states of control and target systems become *entangled*

$$C[|\psi\rangle \otimes |d\rangle] = \sum_k \psi_k [|k\rangle \otimes (U_k|d\rangle)]. \quad (4)$$

The *programmable quantum state transfer* is defined in present work as a quantum network ensuring Eqs. (1, 2) with *spatially distributed system as a target*.

It should be mentioned, that any universal set of quantum gates with distributed quantum systems might include possibility of information transfer, e.g., quantum interfaces [4] intensively use controlled gates with different purposes. So it is justifiable to restrict discussion

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here to more specialized class of quantum networks described by Eq. (1) *displaying a correspondence principle with standard transport models like lattice gases and random walks.*

### III. QUANTUM BOTS ON LATTICES

Perfect cloning of arbitrary unknown quantum states is forbidden [5] and we may limit consideration to transmission without proliferation. A simple model of such a transfer is a linear lattice with  $N$  sites, two-dimensional control space, and evolution described by Eq. (1) with  $U_0 = U_1^* = U$ , where  $U$  is the shift operator

$$B = |0\rangle\langle 0| \otimes U + |1\rangle\langle 1| \otimes U^*, \quad U_{ij} = \delta_{i,j+1 \bmod N}. \quad (5)$$

It may be considered as a rudimentary version of a quantum robot [6], and a term “quantum bot” or “qubot” was suggested for such a system [7, 8].

In fact, Eq. (5) corresponds to quantum mechanical notation for a simplest example of one-dimensional invertible *lattice gas cellular automaton* (LGCA) [9] with “control bit” denoting direction of motion. A similar model is also known in the theory of quantum cellular automata [10] and coined quantum walks [11]. The Eq. (5) corresponds to cyclic boundary condition and similar quantum extensions for boundaries with reflections and multidimensional lattices may be also constructed using classical LGCA [7, 8, 9, 10].

It is possible to quantize the “almost classical” model of the state transfer Eq. (5) and adapt it for quantum control or computations with higher dimensional systems [12, 13]. For example, in the quantum case it is possible to consider “ $n$ -th roots” of the operator  $U$  with properties  $R^n = U$  and a related question about *continuous generalization* of this discrete time model using the limit  $n \rightarrow \infty$  and even about the Hamiltonian of such evolution.

There is quite straightforward approach to such a question [7] using diagonalization of  $U$  by the discrete Fourier transform  $F_{jk} = e^{2\pi i j k / N} / \sqrt{N}$ , *i.e.*, a discrete analogue of transition between coordinate and momenta spaces

$$U = F^* V F, \quad V_{kj} = e^{2\pi i k / N} \delta_{kj}. \quad (6)$$

The matrices  $V$  and  $U$  form so-called Weyl pair and in continuous limit  $U$  corresponds to translation operator  $\exp(i\tau p) : \psi(x) \rightarrow \psi(x + \tau)$  [14].

Due to Eq. (6) it is possible to suggest an expression for family of “roots”  $U(\alpha)$ ,

$$U(\alpha) = F^* V(\alpha) F, \quad V_{kj}(\alpha) = e^{2\pi i \alpha k / N} \delta_{kj} \quad (7)$$

where  $U(\alpha)U(\beta) = U(\alpha + \beta)$ ,  $U(1) = U$  and  $R^n = U$  for  $R = U(1/n)$ . A family of operators  $U(\alpha)$  and  $V(\beta)$  resembles *the Weyl system* for continuous case [15], but satisfy necessary Weyl commutation relations only for an integer  $\alpha, \beta$  and should be discussed elsewhere. It is also possible to suggest a Hamiltonian  $H_U$  for the gate  $U$

$$H_U = F^* K F, \quad K_{kj} = 2\pi k \delta_{kj} / N, \quad e^{iH_U t} = U(t). \quad (8)$$

Hamiltonians for networks with more general topology are discussed in [16]. Coefficients of the matrices Eq. (7) and the Hamiltonian Eq. (8) may be simply calculated directly [7, 17]

$$U_{jk}(\alpha) = \hat{s}_N(k - j + \alpha), \quad \hat{s}_N(x) \equiv \frac{e^{i\varpi x} \sin(\pi x)}{N \sin(\pi x / N)}, \quad (9)$$

where  $\varpi \equiv \pi(N - 1)/N$ ,

$$(H_U)_{jj} = \varpi, \quad (H_U)_{jk} = \frac{2\pi/N}{1 - e^{2\pi i(k-j)/N}}, \quad j \neq k. \quad (10)$$

Using Eq. (8), a Hamiltonian of the conditional evolution Eq. (5) may be represented as

$$H_{\perp} = |0\rangle\langle 0| \otimes H_U - |1\rangle\langle 1| \otimes H_U = \sigma_z \otimes H_U. \quad (11)$$

Similar models may switch over different outputs in according to a state of control. Let us consider two-dimensional control space  $\mathcal{H}_P$ , two coupled lattices with Hilbert space  $\mathcal{H}_{2N} = \mathcal{H}_2 \otimes \mathcal{H}_N$  as a target, and the Hamiltonian

$$H_{\pm} = \mathbb{1} \otimes (\mathbb{1} \otimes H_U) + \frac{2}{N-1} |1\rangle\langle 1| \otimes (\sigma_x \otimes \mathbb{1}) \quad (12)$$

on  $\mathcal{H}_P \otimes (\mathcal{H}_2 \otimes \mathcal{H}_N)$ . It corresponds to transmission along one lattice for state  $|0\rangle$  of control qubit with additional switch between two lattices for  $|1\rangle$ .

The Eq. (12) is a degenerated case of “chessman” Hamiltonian for two-dimensional  $m \times n$  lattice [7], *i.e.*,  $H_* = \sum_{k,j} |k,j\rangle\langle k,j| \otimes (k\mathbb{1} \otimes H_U + jH_U \otimes \mathbb{1})$ , where  $(k,j)$  are directions of moves.

The Eq. (10) describes a Hamiltonian with long-range interaction and the attenuation law approximately proportional to  $|j - k|^{-1}$  for  $N \gg |j - k|$ . Such a Hamiltonian may produce some problem with precise experimental engineering. It would be good to find an equivalent Hamiltonian with nearest-neighbour interaction and it is discussed in the next section.

The Eq. (9) shows, that only for integer  $\alpha = n$  state is localized  $U(n)|0\rangle = |n\rangle$ , contrary to a nonlocal distribution  $U(\alpha)|0\rangle = \sum_k \hat{s}_N(\alpha - k)|k\rangle$  for real  $\alpha$ .

### IV. PERFECT STATE TRANSFER

In [18] was suggested a Hamiltonian for the quantum spin chain for the perfect state transfer. Let us show, that up to change of basis it produces the same evolution as  $H_U$  in Eq. (8). It is known [19], that the Hamiltonian for a Heisenberg chain used in [18] is immediately related with a Hamiltonian for one particle on a lattice with tunnelling between neighbor sites, *i.e.*, with a basic model of the present paper.

Similar lattice and graph analogues of the chain networks are also well known in the quantum information science [22]. Here is suggested for simplicity, that for the transport of a qubit state is used dual-rail encoding

[18, 23], because such a case has more direct relation with lattice models used here.

The operator  $K$  in Eq. (8) is equal to  $J_z/h + \varpi \mathbb{1}$ , where  $J_z$  corresponds to a “fictitious” particle with spin  $(N-1)/2$ ,  $h$  is Plank’s constant, and  $\varpi$  was introduced after Eq. (9). Operators  $J_x$ ,  $J_z = (K - \varpi \mathbb{1})h$  and  $H'_U = (H_U - \varpi \mathbb{1})h$  have the same eigenvalues and there is some operator  $O_x$ :  $O_x H'_U O_x^* = J_x$ . It is enough to use the composition of the Fourier transform and the transition between  $J_x$  and  $J_z$  basis (see [19, 20]).

The operator  $\Omega J_x$  with a strength parameter  $\Omega$  corresponds to the Hamiltonian of the *perfect state transfer* introduced in [18] and resolves the problem with a “nearest-neighbour representation” of the Hamiltonian  $H_U$  for the shift operator. In this basis instead of the shift matrix  $U$  we have higher dimensional representation of a rotation  $R_x(t) = \exp(itJ_x/h)$  [18, 19, 20, 21]. Unlike the operator  $\exp(iH_U t) = U(t)$ , it displays localization of the initial state  $|0\rangle$  only for the extreme points of a lattice (chain), but it is enough for the perfect state transfer.

A conditional Hamiltonian like Eq. (11) for such a transfer is  $\sigma_z \otimes J_x$ . More useful is analogue of Eq. (12), i.e.,  $H_\pm = \mathbb{1} \otimes \mathbb{1} \otimes J_x + \frac{2h}{N-1} |1\rangle\langle 1| \otimes \sigma_x \otimes \mathbb{1}$ , controlling of switch between two different output lines. Such Hamiltonians describe a controlled scalar excitation, but for transfer of a qubit state it is enough to double number of lattices.

In general, we can consider such a kind of models as a tensor product of three Hilbert spaces  $\mathcal{H}_S \otimes \mathcal{H}_P \otimes \mathcal{H}_d$ , viz, transmitted state, program and distributed target respectively. It is also can be considered as an extension of a control system to  $\mathcal{H}_S \otimes \mathcal{H}_P$ , when only subsystem  $\mathcal{H}_P$  may affect on transfer by  $\mathcal{H}_d$ .

Most methods discussed here may be used almost without change both for lattices and spin chains, because for the state transfer via spin chains with  $n$  nodes nowadays [18, 21, 23, 24] often is used only  $n$ -dimensional subspace of whole  $2^n$ -dimensional Hilbert space and a lattice with  $n$  nodes may be used instead.

For a spin chain a simple relation with a lattice model exists only for spin-half particles and so, using one lattice with an internal space  $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_P$  for the control and the transferred state, we lose the analogy with spin chains.

It is possible to realize the control with some quantum system attached to a single lattice or use multiple parallel lattices or spin chains [24]. The design with spin chains should utilize some interactions [21, 25] for conditional dynamics. Realistic models of quantum information devices appropriate for such purposes may be found in many papers, from earliest suggestions [26, 27] till more recent works [25, 28].

It should be emphasized, that there is some difference between a model of a global control of such chains [28] and the programmable dynamics discussed in the present paper. It is usual distinction between general and programmable quantum networks [29, 30], between the ex-

ternal control and the transfer driven by an internal state encoding a program of motion.

## V. COINED QUANTUM WALK

A coined quantum walk on a circle [11] may be considered *formally* as a special example of conditional quantum dynamics  $B$  Eq. (5) with a control register, altered on each step by the Hadamard transform  $H = (\mathbb{1} + i\sigma_y)/\sqrt{2}$  (or symmetric analogue  $(\mathbb{1} + i\sigma_x)/\sqrt{2}$  [31]) with  $T$  steps of evolution described by the operator  $(BH)^T$ .

The theory of coined quantum walk has interesting outcome to analysis of the programmable quantum networks, because produces a wide set of examples with feeding a control register by nonorthogonal states. It was mentioned, that for usual theory of (non-stochastic) programmable quantum networks [1, 2, 3, 29, 30] different states of a program should be orthogonal to ensure Eq. (2) and prevent entanglement between the program and a data Eq. (4).

It is convenient also to compare the coined quantum walk with the *programmable quantum processors* [3, 29, 30] containing third system, “a tape” and a gate  $F$  for altering of a state of a control register after each step of evolution. So, instead of one gate  $G$  Eq. (2) is used an analogue of classical processor timing  $(FG)^T$  [3, 29, 30].

A similar design may be used for programmable implementation of a coined quantum walk controlled by altered coin(s). Different models with set of (random) coins provide possibility of “tuning” from the quantum to classical-like behavior [32, 33]. The programmable implementation of such a model could be compared with the generation of (pseudo)random numbers and the Monte-Carlo simulations by a classical computer.

Let us recall a quantum bot  $B$  Eq. (5) with the control register used as a coin space. An application  $(BC_\theta)^T$  with coins like  $C_\theta = \exp(i\theta\sigma_x)$ ,  $\theta \in [0, 2\pi]$  provide a smooth transition between a uniform motion and behavior of quantum walk [34]. It is possible to introduce a simplest controlled coined quantum walk with  $n$  different coins  $U_k$  and three quantum systems with a step composed from two operators:  $S_{(123)} = C_{(12)} B_{(23)}$ .

Here  $C_{(12)} \equiv C \otimes \mathbb{1}$  is  $C$  operator Eq. (1) for first and second systems,  $B_{(23)} \equiv \mathbb{1} \otimes B$  is the operator  $B$  Eq. (5) on second and third systems. If first system has a state  $|k\rangle$  during  $T$  steps,  $(S_{(123)})^T$  is a “qubot driven” quantum walk on third system with a coin  $U_k$ .

A state of a coin may be entangled with a state of a lattice. In such a case the second and third systems may be considered as a joint target, controlled by a state of first system.

Generalization of  $C$  Eq. (1) for continuous parameters may be produced by the simple change of the sum to an integral [29, 30] and let us use smooth tuning of coins like  $C_\theta$  above.

## VI. CONCLUSIONS

In this work was considered unified approach to different models of the programmable quantum state transfer. It was used some methods of construction of programmable quantum networks with a higher-dimensional target system adapted for specific properties of distributed dynamical models.

It was shown, that a simple “qubot” model may be extended to a programmable system associated with a short-range Hamiltonian, coinciding with  $\Omega J_x$  operator for some fictitious particle with high spin and widely used nowadays in the theory of the perfect state transfer.

In the paper was considered only the *coined* quantum

walk because of particular structure. Formally, the coin space resembles a specific version of a control register and so a programmable model of such a system should use a “cascade” with two control registers for a single target system. It may be formally treated also using a joint system with a coin and a lattice as a new target for control.

It is shown also, that the application of the theory of programmable quantum networks illustrates some useful relations between three models mentioned above: the quantum bots and lattice gas automata (see Sec. III), the perfect quantum state transfer (see Sec. IV), and the coined quantum walk (see Sec. V).

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